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Relationship between the Proportional Hazards Model and the Proportional Mean Residual Life Model

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ABSTRACT

Due to its numerous but important applications, the mean residual life (MRL) has seized the curiosity of several researchers both in theory and in practice. It is defined as the remaining life expectancy of a subject given its survival up to time t. As an alternative approach to the Cox proportional hazards (PH) model in studying the relationship of survival times with the subject's explanatory covariates, the proportional mean residual life (PMRL) model was introduced in the literature. These two models are the same in form in that the function of interest is expressed in terms of the product of its baseline function and some proportionality constant. The only difference is that the latter directly models the MRL instead of the hazard. In this paper, the relationship between these two models is investigated analytically and illustrated by actual data. General relationships between the hazard function and mean residual life function are presented as well.

Key words and phrases: proportional mean residual life model, proportional hazards model, life expectancy, hazard function, censoring

I. INTRODUCTION

A major concern in studying survival data is estimating the basic functions associated with survival times. This might be the survival function, hazard function, or the mean residual life function. Formally, when T (representing the time until the occurrence of the event of interest) is a nonnegative random variable on a probability space $(\Omega,\Im,P)$, the hazard function, $h(t)=\lim_{\Delta t\to 0}\frac{P\left(t\le T\le t+\Delta t\mid T\ge t\right)}{\Delta t}$, gives the risk of failure per unit time along the "aging process." On the other hand, the mean residual life function, $m(t)=E\left[T-t\mid T\ge t\right]$, is the remaining life expectancy of a subject given its survival up to time t. While the hazard function at t provides information about a very small interval after t, the MRL provides information about the whole interval after t (Guess and Proschan, 1988).

Many researchers, however, have been enthralled to study the relationship between survival times and their explanatory variables or covariates. Accordingly, Cox introduced a general nonparametric proportional hazards (PH) model, having the hazard function as the response function. When the survival times are continuously distributed, the hazard is

$$h(t\mid \mathbf{Z})=h_0(t)\exp(\gamma'\mathbf{Z}) \tag{1.1}$$

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where **Z** is the p-vector of covariates, γ is the p-vector of regression coefficients, and $h_0(t)$ is the (baseline) hazard function of the underlying survival distribution (arbitrary) when **Z** is ignored, that is, $\mathbf{Z} = \mathbf{0}$ (Lee, 1992).

As an alternative to the Cox proportional hazards model in studying the association of survival times and their covariates, Oakes and Dasu (1990) proposed the proportional mean residual life (PMRL) model:

$$m(t \mid \mathbf{Z}) = m_0(t)\exp(\boldsymbol{\beta}'\mathbf{Z}) \tag{1.2}$$

where $m(t \mid \mathbf{Z})$ is the mean residual life function, **Z** is the p-vector of covariates, $\boldsymbol{\beta}$ is the p-vector of associated coefficients, and $m_0(t)$ is some unknown mean residual life when $\mathbf{Z}=\mathbf{0}$ and is unspecified in its semi-parametric version.

Notice that models (1.1) and (1.2) are the same in form in that the function of interest is expressed in terms of the product of its baseline function and some proportionality constant. Both involve baseline functions where the p-vector of covariates **Z** is **0**, and some proportionality constant expressed as the exponential of a linear combination of the covariates. The only difference is that the first models the hazard rate or risk of immediate failure, while the latter models the mean residual life function or life expectancy. A natural question to ask is: how is the Cox proportional hazards model related to Oakes and Dasu's proportional mean residual life model? Since the two models arise from the same survival times and covariates, what could be the link between their baseline functions and how are their parameters related?

Oakes and Dasu (1990) and Gupta and Kirmani (1998) were able to show that the two models coincide if and only if $m_0(t)$ is linear in t. In addition to this, its underlying distributions then belong to the Hall-Wellner class of distributions. That is, its baseline MRL function is of the form $m_0(t) = At + B$, where A>-1 and B>0 (Hall and Wellner, 1981). This yields the Pareto, Exponential, and rescaled Beta distribution when A>0, A=0, and -1<A<0 respectively (Oakes and Dasu, 1990). Meanwhile, Guess and Proschan (1988) noted that the MRL is inversely related to the hazard function corresponding to the residual life time of stationary renewal process.

In this paper, we shall find the relationship of the PH model and the PMRL model by utilizing well known mathematical relationships between the survival function, hazard function, and the mean residual life function. Specifically, this study intends to analytically find the relationship of the baseline hazard function $h_0(t)$ and the baseline mean residual life function $m_0(t)$, and hopes to ferret the connection between the parameters of the two models. Thus, the next section is first devoted to discuss the correspondence between the basic quantities in survival analysis in general, that is, without assuming models (1) and/or (2). After which, given the p-vector of covariates **Z**, we present the relationship between the conditional mean residual life $m(t|\mathbf{Z})$ and the conditional hazard function $h(t|\mathbf{Z})$ under the assumption of proportionality of the hazards and proportionality of the mean residual life separately in Section 3, and then investigate further by assuming proportionality of both the hazards and mean residual life in Section 4. After which, we shall illustrate the results analytically and empirically in Sections 5 and 6 respectively, then conclude with Section 7.

II. GENERAL RELATIONSHIP BETWEEN HAZARD AND MEAN RESIDUAL LIFE

Let us first recall that if T is a nonnegative random variable representing the failure time with density $f(t)$, the mean residual life function is

$$m(t) = E[T - t \mid T > t] = \begin{cases} \dfrac{\int_t^{\infty} x f(x)dx}{S(t)} - t & , for\ S(t) > 0 \\ 0 & , for\ S(t) = 0 \end{cases}$$

Note that if $S(0) = 1$, the mean or expected value of T is just the MRL at time zero, i.e., $\mu = E[T] = m(0)$. However, when $S(0) < 1$, $m(0) = \mu/S(0) \neq \mu$. For simplicity, we shall assume that $S(0) = 1$ unless otherwise indicated.

Let us now look into the relationship, in general, of the MRL function with the other basic functions associated with reliability and survival analysis. The following theorem relates the MRL to the survival function and the survival function to the MRL:

Theorem 1: Let T be a nonnegative random variable with survival function $S(t)$ and density $f(t) > 0$ such that $E[T] < \infty$, and let $\tau = \sup\{t \mid S(t) > 0\}$. Then for $0 \leq t \leq \tau$,

1(a). $m(t) = \dfrac{\int_t^{\infty} S(x)dx}{S(t)}$; and

1(b). $S(t) = \dfrac{m(0)}{m(t)} \exp\left\{ -\int_0^t \dfrac{1}{m(x)} dx \right\}$.

Proof:

We furnish you with a proof of (a) as that of Klein & Moeschberger, (1997). By definition, we have

$$m(t) = E[T - t \mid T > t] = \frac{\int_t^{\infty} (x - t) f(x)dx}{S(t)}$$

for $0 \leq t \leq \tau$. Integrating by parts (letting $u = (x-t)$ and $dv = f(x)dx$ and using the fact that $f(x) = -dS(x)/dx$) yield

$$E[T - t \mid T > t]S(t) = -(x - t)S(x)\Big|_t^{\infty} + \int_t^{\infty} S(x)dx.$$

Noting that $\lim_{x\to\infty} S(x)=0$ and $\frac{d}{dx}S(x)=-f(x)$, the first term in the right-hand side of the above equation is zero because

$$\begin{aligned}\lim_{x\to\infty}\left[-(x-t)S(x)\right] &= -\lim_{x\to\infty}\frac{x}{\frac{1}{S(x)}} = -\lim_{x\to\infty}\frac{[S(x)]^2}{f(x)}, \text{ by L'Hopital's rule} \\ &= 0 \text{, since } f(x)>0.\end{aligned}$$

Dividing both sides by S(t) yields the desired result.

By writing (x – t) as $\int_t^x du$, and using Tonelli's formula, an alternative proof is given by Gupta & Bradely (2003).

Let us now supply a direct proof of result (b) by first noting that for $0 \le t \le \tau$, $S(t) > 0$ so that $m(t) > 0$ as well. Hence we can rewrite (a) as

$$\frac{1}{m(t)} = \frac{S(t)}{\int_t^\infty S(x)dx}.$$

But since $\frac{1}{m(t)} = \frac{d}{dt}\int_0^t \frac{1}{m(x)}dx$ and $\frac{-S(t)}{\int_t^\infty S(x)dx} = \frac{d}{dt}\left[\log\left(\int_t^\infty S(x)dx\right) - \log\left[m(0)\right]\right]$, we have

$$-\int_0^t \frac{1}{m(x)}dx = \log\left(\int_t^\infty S(x)dx\right) - \log\left[m(0)\right] = \log\left[\frac{\int_t^\infty S(x)dx}{m(0)}\right].$$

This implies that $\int_t^\infty S(x)dx = m(0)\exp\left\{-\int_0^t \frac{1}{m(x)}dx\right\}$. Dividing both sides by $m(t)$ and multiplying the left-hand side by $S(t)/S(t)$ completes the proof. ∎

Theorem 1(a) expresses the MRL in terms of the survival function. This gives us a geometric interpretation of the MRL, i.e., $m(t)$ is the area under the survival curve to the right of t divided by $S(t)$.

On the other hand, the inversion formula, Theorem 1(b), expresses the survival function in terms of the MRL. This is often presented in the literature, but among those which we have encountered, however, only Meilijson (1972) provided a proof. His interesting approach is different from what we have just presented in that he recognized the relationship of the distribution of the residual life of a stationary renewal process with the MRL.

Theorem 1 shows that there is a one-to-one correspondence between survival functions (with finite means) and MRL functions (Chen, Hollander and Langberg, 1983).

Hence, the MRL function uniquely determines the survival distribution so that it can also be used for modeling (Hall and Wellner, 1981).

We now utilize Theorem 1 to relate the MRL with the density function, hazard function, and cumulative hazard function.

Theorem 2: Under the same assumptions as in Theorem 1,

2(a). $f(t)=\left(\frac{d}{dt}m(t)+1\right)\left(\frac{m(0)}{m(t)^2}\right)\exp\left\{-\int_0^t\frac{1}{m(x)}dx\right\}$;

2(b). $h(t)=\left(\frac{d}{dt}m(t)+1\right)\Big/m(t)$; and

2©. $m(t)=\int_0^\infty \exp\{H(t)-H(t+x)\}dx$.

Proof:

It is straightforward to show Theorem 2(a) by again using the relationship $f(x)=-dS(x)/dx$.

To show Theorem 2(b), it is well known that $h(t) = f(t)/S(t)$. Substituting the expression in Theorem 2(a) and in Theorem1(b) correspondingly and simplifying will yield

$$h(t)=\frac{\left(\frac{d}{dt}m(t)+1\right)}{m(t)}$$

Alternatively, one can exploit the fact that $h(t) = -d[\log S(t)]/dt$ and apply it to Theorem1(a).

Now we utilize the fact that $S(t)=\exp\left\{-\int_0^t h(u)du\right\}=\exp\{-H(t)\}$ to prove Theorem 2(c). Letting $u=x-t$ in Theorem 1(a), we have

$$m(t)=\int_0^\infty\frac{S(t+u)}{S(t)}du=\int_0^\infty\frac{\exp\{-H(t+u)\}}{\exp\{-H(t)\}}du$$

Therefore, $m(t)=\int_0^\infty \exp\{H(t)-H(t+x)\}dx$.

An alternative derivation of Theorem 2(c) can be found in Gupta & Bradely (2003) ■

Theorem 2(b) poses a natural restriction on the rate of change of the MRL, $\frac{d}{dt}m(t)$. Since $h(t)>0$, $\left(\frac{d}{dt}m(t)+1\right)/m(t)>0$. This in turn implies that $\frac{d}{dt}m(t)>-1$.

Although rarely found in books, Theorem 2(b) is popular among papers which investigate the MRL in conjunction with the hazard because it shows the relationship between the two.

We now have expressions of the MRL in terms of the survival function and the cumulative hazard function, and expressions of the survival function, density function, and hazard function in terms of the MRL. Ideally however, we would like to express the MRL in terms of the hazard rate or some of its known functions or derivatives without the complication of integrals. Moreover, note that these equations do not involve covariates yet. For the intention of investigating the relationship between the proportional mean residual life (PMRL) model and the proportional hazards (PH) model, we extend these results naturally to include covariates as follows:

1. $m(t \mid \mathbf{Z}) = \dfrac{\int_t^{\infty} S(x \mid \mathbf{Z})dx}{S(t \mid \mathbf{Z})}$;

2. $m(t \mid \mathbf{Z}) = \int_0^{\infty} \exp\{H(t \mid \mathbf{Z}) - H(t+x \mid \mathbf{Z})\}dx$;

3. $S(t \mid \mathbf{Z}) = \dfrac{m(0 \mid \mathbf{Z})}{m(t \mid \mathbf{Z})} \exp\left\{-\int_0^t \dfrac{1}{m(x \mid \mathbf{Z})}dx\right\}$;

4. $f(t \mid \mathbf{Z}) = \left(\dfrac{d}{dt}m(t \mid \mathbf{Z}) + 1\right)\left(\dfrac{m(0 \mid \mathbf{Z})}{m(t \mid \mathbf{Z})^2}\right)\exp\left\{-\int_0^t \dfrac{1}{m(x \mid \mathbf{Z})}dx\right\}$; and

5. $h(t \mid \mathbf{Z}) = \left(\dfrac{d}{dt}m(t \mid \mathbf{Z}) + 1\right)\Big/ m(t \mid \mathbf{Z})$.

III. HAZARD AND MEAN RESIDUAL LIFE UNDER THE PROPORTIONAL HAZARDS OR PROPORTIONAL MEAN RESIDUAL LIFE ASSUMPTION

Cox's PH model has been widely used and studied in practice and in the literature. Its theoretical and practical characterization has been virtually present in any survival analysis textbook. The next theorem recollects the relationships that exist between the basic functions associated with survival analysis under the PH model.

Theorem 3: Under the proportional hazards model assumption, the following equalities hold:

3(*a*). $S(t \mid \mathbf{Z}) = S_0(t)^{\exp\{\gamma'\mathbf{Z}\}}$, where $S_0(t) = \exp\left\{\int_0^t h_0(x)dx\right\}$;

3(*b*). $H(t \mid \mathbf{Z}) = H_0(t)\exp\{\gamma'\mathbf{Z}\}$, where $H_0(t) = \int_0^t h_0(x)dx$;

3(*c*). $f(t \mid \mathbf{Z}) = \exp\{\gamma'\mathbf{Z}\} f_0(t)\left[\int_t^{\infty} f_0(x)dx\right]^{\exp\{\gamma'\mathbf{Z}\}-1}$, where $f_0(t) = h_0(t)S_0(t)$; and

$$3(\boldsymbol{d}).\ m(t\mid \mathbf{Z}) = \frac{\int_t^{\infty} \exp\left\{\int_0^x h_0(u)du\right\}^{\exp\{\gamma'\mathbf{Z}\}} dx}{\exp\left\{\int_0^x h_0(u)du\right\}^{\exp\{\gamma'\mathbf{Z}\}}} = \frac{\int_t^{\infty} S_0(x)^{\exp\{\gamma'\mathbf{Z}\}} dx}{S_0(x)^{\exp\{\gamma'\mathbf{Z}\}}}.$$

Together, these formulas provide the rudimentary theoretical characterization of the PH model. Their proofs are relatively simple via straightforward deployment of fundamental relationships between survival, hazard, density, and MRL functions. But the importance of these formulas lies in being able to relate the basic survival analysis functions conditioned on some covariates with their baseline counterparts.

Shifting now to the next model of interest, we assume proportionality of the MRL.

Theorem 4: Under the proportional mean residual life model assumption, the following equalities hold:

4(a). Let $S_0(t) = \left(\frac{\mu_0}{m_0(t)}\right)\exp\left\{-\int_0^t \frac{dx}{m_0(x)}\right\}$, where $\mu_0 = m_0(0)$ is the baseline mean lifetime. Then

$$(\boldsymbol{i}).\ S(t\mid \mathbf{Z}) = S_0(t)\left(\frac{\int_t^{\infty} S_0(x)dx}{\mu_0}\right)^{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1};$$

$$(\boldsymbol{ii}).\ S(t\mid \mathbf{Z}) = S_0(t)^{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}}\left(\frac{m_0(t)}{\mu_0}\right)^{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1};\ \text{and}$$

$$4(\boldsymbol{b}).\ h(t\mid \mathbf{Z}) = \frac{\frac{d}{dt}m_0(t)\exp\{\boldsymbol{\beta}'\mathbf{Z}\}+1}{m_0(t)\exp\{\boldsymbol{\beta}'\mathbf{Z}\}} = \frac{\frac{d}{dt}m_0(t)+\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}}{m_0(t)}.$$

The proof of Theorem 4, though not presented, is a direct application of the inversion formula in Theorem 1(b) and of Theorem 2(b). Meanwhile, notice that Theorem 4(a) shows the form of the survival function under model (1.2). Interestingly, Oakes and Dasu (1990) also presented Theorem 4(a.i) except that it was under the two-sample case. Here, the survival function of the model is expressed in terms of the baseline survival function and baseline mean lifetime. A minor complication however, is that, it involves the integral of the baseline survival function. When the baseline MRL is known, an alternative and more convenient expression is Theorem 4(a.ii).

IV. HAZARD AND MEAN RESIDUAL LIFE UNDER BOTH PROPORTIONAL HAZARDS AND PROPORTIONAL MEAN RESIDUAL LIFE ASSUMPTIONS

Let us now investigate the relationship between the two models analytically by illustration using actual data. This time, we assume proportionality of both the hazard and the MRL. The next theorem overtly relates the baseline hazard function with the baseline MRL function.

Theorem 5: The distribution of a nonnegative random variable T satisfies both proportional hazards and proportional mean residual life assumption if and only if the baseline mean residual life function is inversely proportional to the baseline hazard function. That is,

$$h_0(t)m_0(t) = c\text{, where } c = \frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{\exp\{\boldsymbol{\gamma}'\mathbf{Z}\}-1} \text{ is the proportionality constant.}$$

Proof:

Suppose both $(i)\, h(t\,|\,\mathbf{Z}) = h_0(t)\exp(\boldsymbol{\gamma}'\mathbf{Z})$ and $(ii)\; m(t\,|\,\mathbf{Z}) = m_0(t)\exp(\boldsymbol{\beta}'\mathbf{Z})$ are satisfied. Then (ii) and Theorem 4(b) implies

$$h(t\,|\,\mathbf{Z}) = h_0(t) + \frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{m_0(t)}.$$

But under (i), $h(t\,|\,\mathbf{Z}) = h_0(t)\exp(\boldsymbol{\gamma}'\mathbf{Z})$, and so

$$h_0(t)\exp\{\boldsymbol{\gamma}'\mathbf{Z}\} - h_0(t) = \frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{m_0(t)}.$$

Factoring out $h_0(t)$ in the left hand side and doing a little more algebra yields

$$h_0(t)m_0(t) = \frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{\exp\{\boldsymbol{\gamma}'\mathbf{Z}\}-1} = c,$$

where c is constant since $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are vector of constants and $\mathbf{Z}$ is known. ■

Theorem 5 might initially lead us to think that if we know that the baseline distribution satisfies both PH and PMRL model, then we can always find a straightforward relationship between the parameters $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ by equating the product of the baseline functions with $\frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{\exp\{\boldsymbol{\gamma}'\mathbf{Z}\}-1}$ then express one parameter in terms of the other. This might not always be possible, though. But in the presence of a single covariate, that is $Z \in \Re^1$, it is straightforward to see that

$$\beta = \frac{-\log\{c[\exp(\gamma Z)-1]+1\}}{Z} \tag{4.1}$$

The next theorem characterizes the relationship between the baseline hazard and baseline MRL under the Hall-Wellner family of distributions.

Theorem 6: $m_0(t) = At + B$, where *A*>-1 and *B*>0, if and only if $h_0(t)m_0(t) = c$ for some constant *c*>0.

Proof:

Suppose $m_0(t) = At + B$, with *A*>-1 and *B*>0. Then from Theorem 2(b),

$$h_0(t) = \frac{\frac{d}{dt}m_0(t)+1}{m_0(t)} = \frac{A+1}{At+B}.$$

Hence, $h_0(t)m_0(t) = A+1 = c$.

Now suppose $h_0(t)m_0(t) = c$. Then $\frac{d}{dt}m_0(t) = c-1$, again by Theorem 2(b). That is, $\frac{d}{dt}m_0(t)$ is constant. This happens if and only if $m_0(t)$ is linear in *t*. ■

Theorem 6 states that the baseline distribution is a member of the Hall-Wellner family (distributions having MRL linear in *t*) if and only if the baseline hazard function is inversely proportional to the baseline MRL function. Hence, if we want to find out whether a set of observations spawned from a member of the Hall-Wellner family, we can graph the product of the estimated baseline hazard and baseline MRL and examine its tendency to be a flat curve.

Theorems 5 and 6, together, imply that a distribution satisfying both proportional hazards and proportional MRL belongs to the Hall-Wellner class of distributions. This provides an alternative proof for Oakes and Dasu's Theorem 2 (1990). Moreover, if one is interested in the slope or the rate of change of the baseline MRL, Theorems 5 and 6 jointly give

$$A \doteq \frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{\exp\{\boldsymbol{\gamma}'\mathbf{Z}\}-1} - 1. \tag{4.2}$$

Therefore for a particular dataset realized from a member of the Hall-Wellner class of distributions, we can estimate the slope of the baseline MRL by substituting the regression parameter estimates of the PH and PMRL model in equation 4.2.

The expression of the hazard function in terms of the MRL function in Theorem 2(b) has been so far, of great utility in the previous discussions. Until now, however, we were not able to explicitly express the MRL in terms of the hazard. Nevertheless, if we assume that the MRL is linear in *t*, i.e, $m(t) = At + B$, we have

$$m(t) = \frac{A+1}{h(t)}. \tag{4.3}$$

V. ANALYTIC EXAMPLES

In this section, we demonstrate the results obtained in the previous section by considering the Exponential distribution and the shifted Pareto distribution. After which, the next section would show how they work in reality using the well-known Veterans' Administration (VA) lung cancer trial.

Example 1: Exponential Baseline Distribution

Suppose the baseline distribution is Exponential with parameter $\lambda > 0$. The baseline hazard is then

$$h_0(t) = \frac{f_0(t)}{S_0(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \tag{5.1}$$

and the baseline MRL function is

$$m_0(t) = \frac{\int_t^{\infty} e^{-\lambda x}dx}{e^{-\lambda t}} = \frac{0-(-\lambda^{-1}e^{-\lambda t})}{e^{-\lambda t}} = \lambda^{-1} \tag{5.2}$$

Hence, $h_0(t)m_0(t) = c$, where $c = 1$. It follows from Theorem 5 that

$$\boldsymbol{\beta} = -\boldsymbol{\gamma}. \tag{5.3}$$

This implies that estimating $\boldsymbol{\gamma}$ under Cox's PH model is equivalent to estimating $\boldsymbol{\beta}$ under Oakes and Dasu's PMRL model if the underlying distribution is exponential. In real life survival data however, this is rarely the case. But in addition, equation 5.3 also means that testing the null hypothesis that $\boldsymbol{\beta}+\boldsymbol{\gamma}=0$ is equivalent to testing the hypothesis that the baseline distribution is exponential. More generally, one can test for exponentiality if the product of the baseline hazard and baseline MRL is unity for all t.

It is nice that for the exponential baseline distribution, a direct relationship between $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ exists. However, this does not always happen as in the next example.

Example 2: Shifted Pareto Baseline Distribution

Consider the Pareto Distribution with distribution function $F_X(x) = 1-\left(\frac{b}{x}\right)^a$ where a and b are both greater than zero which is also used to model lifetimes. However, notice that the distribution function is defined only for $x \geq b$. We do not want this restriction for lifetimes. Thus, we let $T = X - b$ so that $t \geq 0$. The distribution function of T is then

$$F_T(t) = 1-\left(\frac{b}{t+b}\right)^a \tag{5.4}$$

Let us now assume that the baseline distribution is as stated in equation 5.4. Then the baseline survival function is $S_0(t)=\left(\frac{b}{t+b}\right)^a$, the hazard function is

$$h_0(t)=\frac{f_0(t)}{S_0(t)}=\left(\frac{ab^a}{(t+b)^{a+1}}\right)\left(\frac{b}{t+b}\right)^{-a}=\frac{ab^a b^{-a}}{(t+b)^{a+1}(t+b)^{-a}}=\frac{a}{t+b} \qquad (5.5)$$

and the MRL function is

$$m_0(t)=\frac{\int_t^\infty \left(\frac{b}{x+b}\right)^a dx}{\left(\frac{b}{t+b}\right)^a}=\frac{t+b}{a-1} \quad \text{for } a>1. \qquad (5.6)$$

Equations 5.6 and 5.6 yield $h_0(t)m_0(t)=\frac{a}{(a-1)}=c$. Equating $\frac{a}{(a-1)}$ with $\frac{\exp\{-\boldsymbol{\beta}'\mathbf{Z}\}-1}{\exp(\boldsymbol{\gamma}'\mathbf{Z})-1}$ does not show explicit relationship between $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ however.

One might observe that the baseline distributions in examples 1 and 2 belong to the Hall-Wellner family. Notice that equations 5.2 and 5.6 are linear with $A=0, B=\lambda^{-1}$ and $A=(a-1)^{-1}$, $B=b(a-1)^{-1}$ respectively. This illustrates Theorem 6.

VI. THE VETERANS' ADMINISTRATION LUNG CANCER TRIAL

The well-known Veterans' Administration lung cancer trial (Kalbfleisch and Prentice, 1980) is used as actual data illustration. This dataset has been analyzed by several authors already, but we shall only present the estimates produced by Chen and Cheng (2005) to illustrate theorems 5 and 6 discussed in the previous section. They utilized the no-prior-therapy subgroup which consists of 97 patients, having survival times ranging from 1 to 587 days with 6 censored observations (see Table 6.1).

The Kaplan-Meier estimates of the survival times are shown in Figure 6.1. On the other hand, the mean residual life estimates are displayed in Figure 6.2, and a smooth line is fitted and superimposed to espouse visualization. These values were calculated by means of the Kaplan-Meier analogue of the empirical MRL which was instigated by Chen, Hollander and Langberg in 1979. It is fortunate that for this dataset, the last observation was uncensored. Otherwise, it would be impossible to estimate the MRL unless some assumptions are made. Figure 6.1 reveals an initial impetuous descent in the survival probability estimates up to about time 160 days, but then gradually decreases subsequently. Meanwhile, the MRL in Figure 6.2 generally tends to increase until 160 days then inclined to decrease afterward. It is thus likely that this belongs to the increasing-then-decreasing mean residual life (IDMRL) type of distributions with change point of around 160 days.

Chen and Cheng (2005) considered two covariates — performance status and tumor type. The latter, although having four levels (large, adeno, small, squamous), was treated as categorical with the large type being the reference group. This being the case, he estimated a 4-vector parameter of regression coefficients: that of (1) the performance status, (2) adeno versus large, (3) small versus large, and (4) squamous versus large.

Table 6.1: Summary Statistics for the Lung Cancer Data

Quartile Estimates

Percent	Point Estimate	95% Confidence Interval [Lower	Upper)
75	144.000	117.000	228.000
50	80.000	52.000	105.000
25	29.000	21.000	45.000

Mean	Standard Error
120.739	13.752

Total	Failed	Censored	Percent Censored
97	91	6	6.19

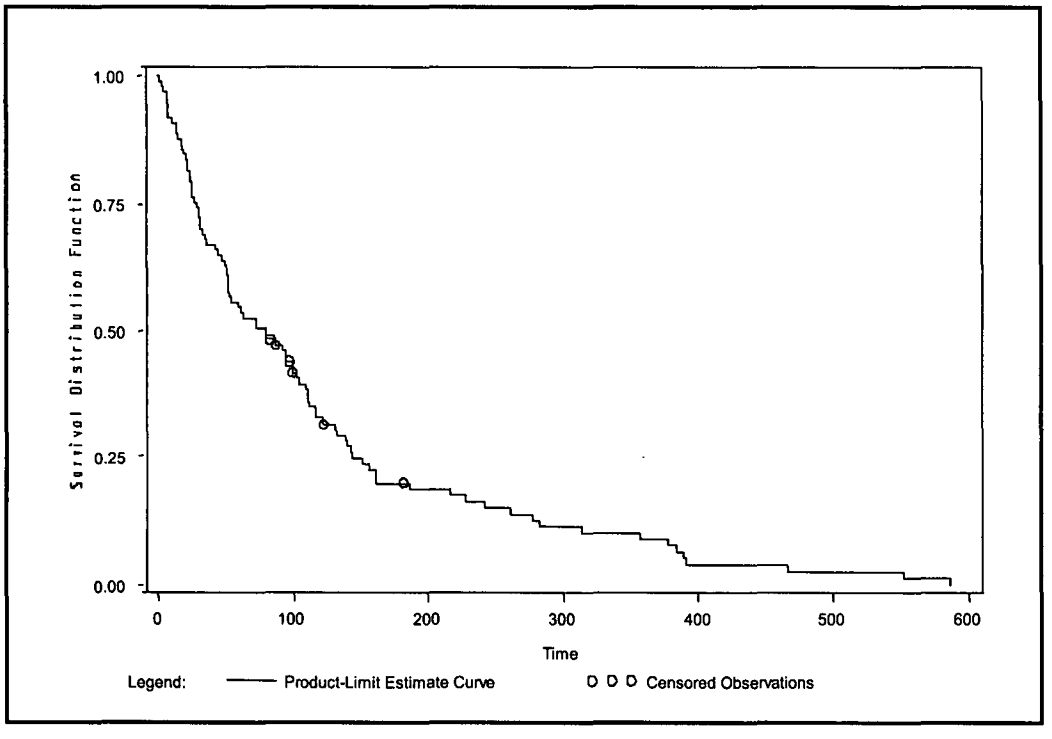

Figure 6.1: Kaplan-Meier Estimates of the Survival Function for the Lung Cancer Data

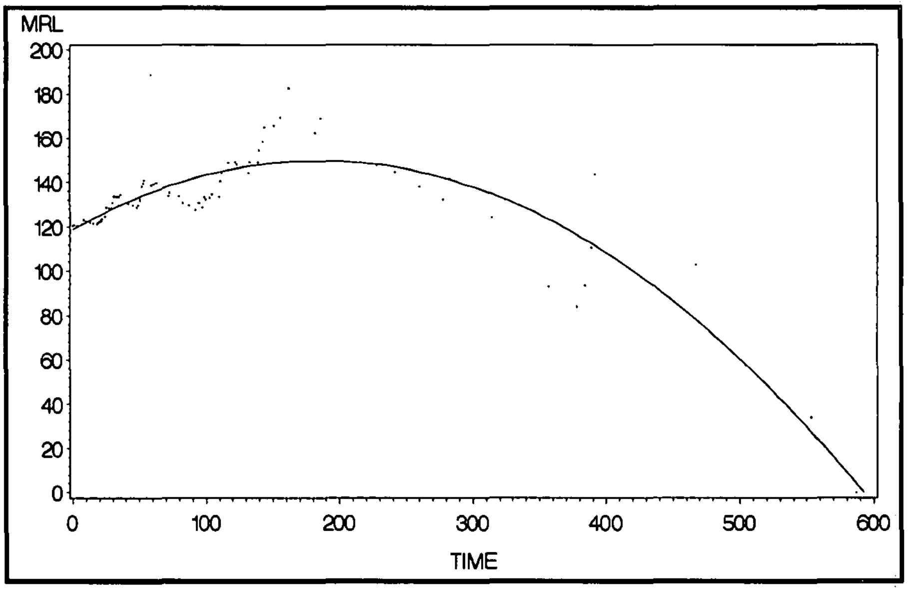

Figure 6.2: The Estimated Mean Residual Life for the Lung Cancer Data (Based on the Kaplan-Meier Analogue of the Empirical Mean Residual Life)

Table 6.2 shows the parameter estimates of Cox's PH model and Oakes and Dasu's PMRL model. Chen and Cheng's (2005) weighted estimation procedure was used to arrive at the values in the last column. Generally, the interpretation of the PMRL parameter estimates is similar to that of Cox's PH model. That is, the average remaining lifetime is estimated to be increased (decreased) by a factor of $\exp(\beta_i)$ for every unit increase in Z_i controlling all other factors. For example in this dataset, the average remaining lifetime of a lung cancer patient is estimated to be increased by a factor of $\exp(0.030) = 1.030$ or 3% for every unit increase in the performance status controlling all other factors.

Going back to Table 6.2, one can notice that the estimates based on Cox's model and those of Oakes and Dasu's model are close to each other in magnitude, but have different signs. This indicates that the two models might be equivalent for this particular data set, and the baseline distribution is possibly exponential as in equation 5.3. To further investigate this, we estimated the baseline survival function using the methodology provided by Kalbfleisch and Prentice (1980), from which the estimates of the baseline hazard function and the baseline MRL function was based. At a glance however, the baseline MRL in Figure 6.3 is clearly not linear, and its product, in Figure 6.4, with the baseline hazard does not exhibit a clear flat curve, which is what should be the case if the underlying distribution is of the Hall-Wellner family by Theorem 5. But, by restricting the window to the first 160 days, one can already imagine a crude linear baseline MRL and a horizontal line in Figure 6.4. This is possibly what brought about the relatively similar parameter magnitudes from the two models, though the baseline MRL is not linear all the way, since the observed lifetimes less than 160 days accounts for 81% of the subjects already. Note however that these observations are drawn by visual inspection only, and hence must not be regarded as concrete. Formal procedures for testing linearity of the baseline MRL still need to be developed to provide stalwart and convincing conclusions.

Table 6.2: Estimates of Regression Coefficient with Standard Errors in the Lung Parenthesis for Cancer Data

Covariates	Cox's PH Model	Oakes and Dasu's PMRL Model
Performance Status	-0.024	0.030
	(0.006)	(0.006)
Tumor Type		
Adeno vs Large	0.851	−0.801
	(0.348)	(0.532)
Small vs Large	0.548	−0.499
	(0.321)	(0.522)
Squamous vs Large	−0.214	0.150
	(0.347)	(0.680)

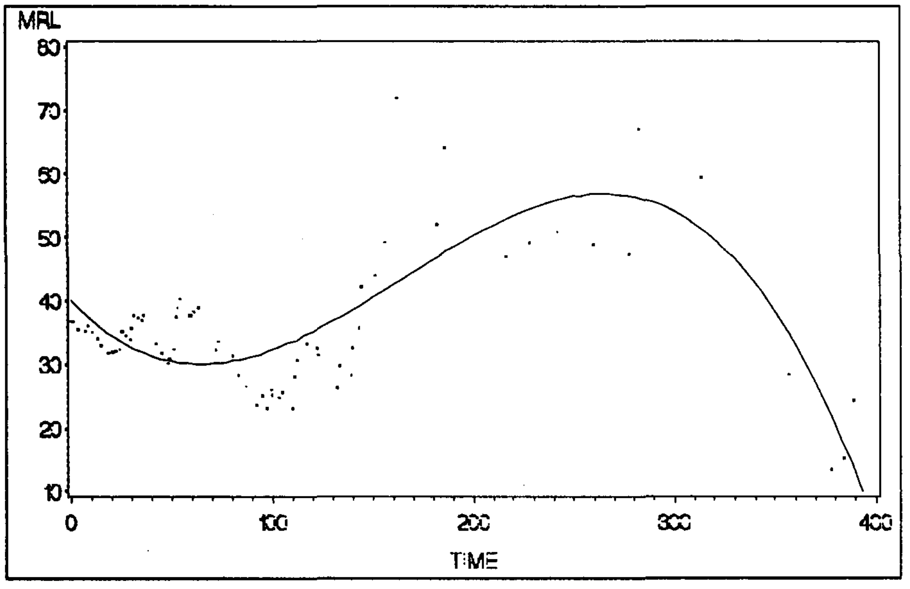

Figure 6.3: The Estimated Baseline Mean Residual Life for the Lung Cancer Data

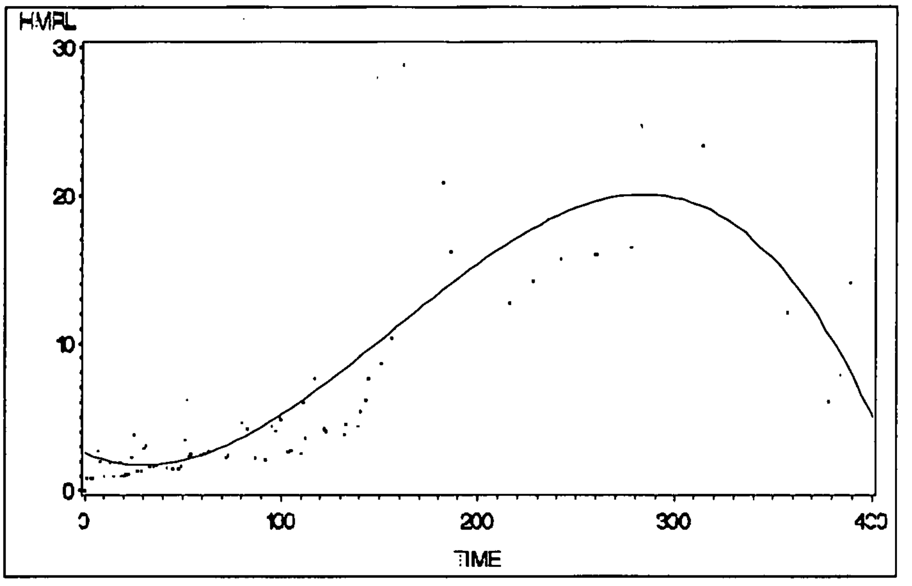

Figure 6.4: Product of the Estimated Baseline Mean Residual Life and the Estimated Baseline Hazard for the Lung Cancer Data

VII SUMMARY AND DISCUSSION

The objective of this paper is to investigate the relationship between Cox's Proportional Hazards (PH) model and Oakes and Dasu's Proportional Mean Residual Life (PMRL) model analytically. It was noted that there is a one-to-one correspondence between the MRL function and the survival function so that the each uniquely determines the other (Theorem 1). But more importantly, we have shown an expression of the hazard in terms of the MRL, though, the presence of the first derivative of the MRL introduces a little hitch (Theorem 2(b)). The converse is not straightforward however. That is, the MRL function can only be expressed in terms of the cumulative hazard function which involves the complication of integrals (Theorem 2(c)). Nevertheless, if we assume that the distribution belongs to the Hall-Wellner family, i.e., the MRL is linear in t, then an explicit expression of the MRL in terms of the hazard can be obtained. Theorems 5 and 6, however, achieved the goal. The former states that a distribution satisfies both PH and PMRL if and only if the baseline MRL is inversely proportional to the baseline hazard, while the latter affirms that the baseline hazard and MRL functions are inversely proportional if and only if the baseline distribution belongs to the Hall-Welner class. This provides an alternative proof for Theorem 2 by Oakes and Dassu (1990), and gives a clear formula for the slope of the underlying linear MRL. On the other hand, we were not able to find a straightforward relationship between their parameters. But if the baseline distribution is exponential, then the product of their baseline hazard and MRL is unity. More so, their parameters, though having opposite signs, are equal in absolute value. Thus, if it is known that the baseline distribution is exponential, modeling using Cox's PH is equivalent to modeling under the PMRL assumption. Shifting from one model to the other will be effortless.

For researchers who are interested in coming up with fresh models, several extensions of the PMRL model can be done. In this paper, we noticed that the covariates were always treated as constants or nonstochastic. A possible modification is to extend Oakes and Dasu's model to accommodate time varying covariates. Meanwhile, to resolve the possibility of an underspecified PMRL model, a frailty variable can be included. Now to accommodate a population with immunes, a PMRL model with immunes can be studied. Likewise, a frailty PMRL model with immunes can be proposed as a generalization of the last two models.

Finally, we surmise that a test of exponentiality can be developed on the basis of Theorem 5. Since the product of the hazard function and the MRL function is unity when the underlying distribution is exponential, testing for exponentiality is equivalent to testing the hypothesis that $m(t)h(t) = 1$. More generally, a test to determine whether empirical survival times spawned from the Hall-Wellner family is possible by examining the flatness of $m(t)h(t)$.

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